Compositional path algebraic structures on directed hypergraphs

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Abstract

We **define** a family of **compositions of n-ary relations** $\overset{(i_k)(j_{k'})}{\underset{n,m}{\circ}}$ parametrised by internal subsequences of entries.

This family is **extended** to a set of operators $\overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}}$ that allow their iteration and are used to define groupoids of tuples, represented as groupoids of hyperedges on a directed hypergraph.

Path algebraic structures are **constructed** from the compositional groupoids already defined.

This construction encompasses the classical path algebra of quivers and, moreover, produces potentially non-associative structures when working in higher arities.

Preliminaries

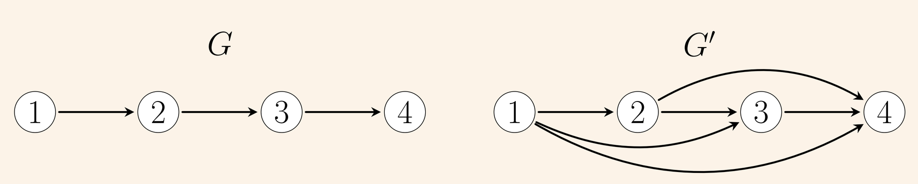
Let V be a finite, non-empty set.

- **Composition of binary relations** [4]: Given $R, S \subseteq V^2$ their composition is defined by:

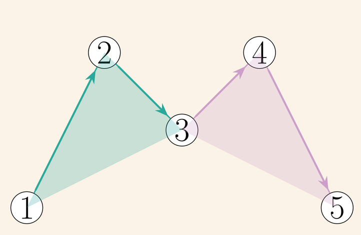
$$R \circ S = \{(x,z) \mid \exists_{y \in V}(\ (x,y) \in R,\ (y,z) \in S)\}$$

- A **simple directed graph** [7] $G = (V, E)$ consists of a set V of vertices, and a finite set $E \subseteq V \times V$ of ordered pairs of vertices called edges.
- The **transitive closure** [7] of a directed graph $G = (V, E)$ is a graph $G' = (V, E')$ where

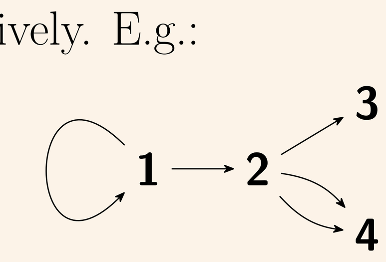
$$E' = \bigcup_{i=1}^{\infty} E^i, \quad E^i = \begin{cases} E, & \text{if } i = 1, \\ E^{i-1} \circ E, & \text{if } i > 1. \end{cases}$$



- **N-ary relations** [2]: Given a set V, an n-ary relation $S \subseteq V^n$ is a subset of the Cartesian product.
- A **directed hypergraph** [3] $H = (V, E)$ consists of a vertex set V and a collection of edges $E \subseteq \bigcup_{k \geq 1} V^k$. Each edge $e \in V^{k_i}$ is an ordered k_i-tuple $(v_1, \ldots, v_{k_i})$. E.g.:



- A **quiver** [6] $Q = (Q_0, Q_1, s, t)$ is defined by a set Q_0 of vertices, a set Q_1 of arrows, and two functions $s, t : Q_1 \to Q_0$ assigning to each arrow its *source* and *target*, respectively. E.g.:



- The **path algebra** [6] kQ of a quiver Q over a field $(k, +, \times)$ is defined as the k-algebra whose vector space has the set of paths of Q as a basis, with multiplication given by concatenation:

$$c \times c' = \begin{cases} cc', & \text{if } t(c) = s(c'), \\ 0, & \text{otherwise.} \end{cases}$$

It is required that for all $\lambda \in k$ and paths c_1, c_2:

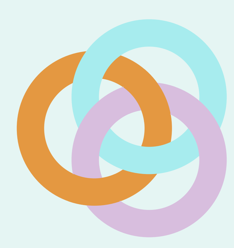
$$\lambda(c_1 c_2) = (\lambda c_1)c_2 = c_1(\lambda c_2) = (c_1 c_2)\lambda.$$

The product of elements of kQ lies in kQ:

$$\sum_{i=1}^{n} \lambda_i c_i \sum_{j=1}^{n} \lambda'_j c'_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda'_j \, c_i c'_j.$$

State of the art

- Composition of n-ary relations: recent definitions for ternary [1] (2021) and n-ary [2] (2024) relations.
- In 2019, Leavitt path algebras on hypergraphs were defined [5], simultaneously generalising the Leavitt algebras of finitely separated graphs and of graphs with weighted vertices and finitely many exits.
- The **higher arity science** paradigm [8] documents higher-order interactions that arise naturally in a wide variety of sciences and complex systems.



The problem

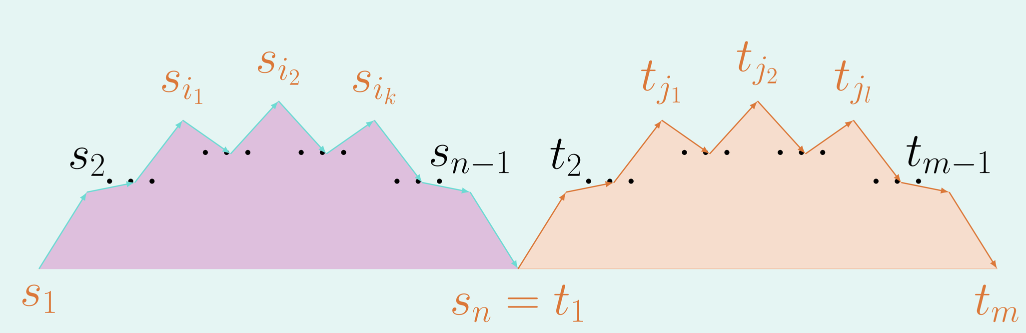
- **Main objective:** Generalise the notion of path algebra by replacing concatenation with composition of n-ary relations and quivers with hypergraphs.
- **Central mathematical problems:**
  1. Define a theoretical framework for studying composition operations on n-ary relations.
  2. Define path algebras that allow modelling non-associative behaviour.
  3. Classify the resulting structures and compare them with classical algebras.
- **Motivation:** N-ary compositions enable the modelling of higher-order interactions (e.g. network operations, high-dimensional data analysis) that are not captured by simple binary paths.

Main results

1. **(n,m)-Compositions** $\overset{(i_k)(j_l)}{\underset{n,m}{\circ}}$.

   Given $s = (s_1, \ldots, s_n) \in V^n, t = (t_1, \ldots, t_m) \in V^m$ and two internal index sequences $(i_r)_{r=1}^{k}$ and $(j_r)_{r=1}^{l}$:

$$\overset{(i_k)(j_{k'})}{\underset{n,m}{\circ}}(s,t) = \begin{cases} \big(s_1, s_{i_1}, \ldots, s_{i_k}, t_{j_1}, \ldots, t_{j_l}, t_m\big), & s_n = t_1, \\ \varnothing, & s_n \neq t_1. \end{cases}$$



   There exist 2^{n+m-4} operators $\overset{(i_k)(j_{k'})}{\underset{n,m}{\circ}}$ for each pair (n, m).

2. **Mixed operator** $\overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}}$ extends $\overset{(i_k)(j_{k'})}{\underset{n,m}{\circ}}$ to chain compositions when the arity changes.
3. **Compositional paths:** chains of hyperedge compositions under a fixed operator $\triangle$.
4. **Compositional path algebra:** kH is the k-algebra whose vector space has the set of compositional paths of H as a basis, with multiplication given by the compositional operator $\overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}}$.

   It is further required that for all $\lambda \in k$ and paths c_1, c_2:

$$\lambda\big(c_1 \overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}} c_2\big) = (\lambda c_1) \overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}} c_2 = c_1 \overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}} (\lambda c_2) = \big(c_1 \overset{(i_k)(j_{k'})}{\underset{n,m}{\triangle}} c_2\big)\lambda$$

   An arbitrary element of kH is: $\sum_{i=1}^{n} \lambda_i c_i$
5. **Theorem:** Every path algebra over a finite quiver is isomorphic to a compositional path algebra on a hypergraph.

Examples

Associative example: polynomial algebra. With a single vertex, i.e. $V = \{1\}$, and a single edge $E = \{(1,1,1)\}$, using $\overset{(2)(2)}{\underset{3,3}{\triangle}}$, the powers a^n generate a compositional path algebra isomorphic to $k[x]$.

$$(1,1,1) \circ (1,1,1) = (1,1,1,1) \mapsto x^2,$$

$$\text{isomorphism } (1,1,1)^n \leftrightarrow x^n.$$

Non-associative example: the compositional path algebra using the operator $\overset{(4,5)(2,3)}{\underset{6,6}{\triangle}}$ on the directed hypergraph with hyperedges $H = \{(1,i,j,k,i,1), (1,j,k,i,j,1), (1,k,i,j,k,1)\}$ turns out to be non-associative.

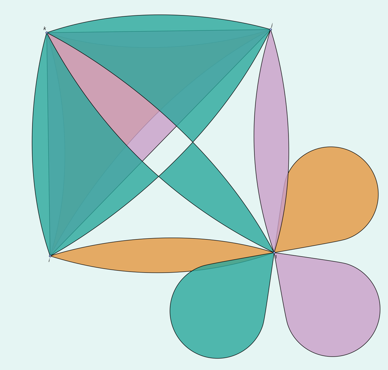


Figure 1: Directed hypergraph.

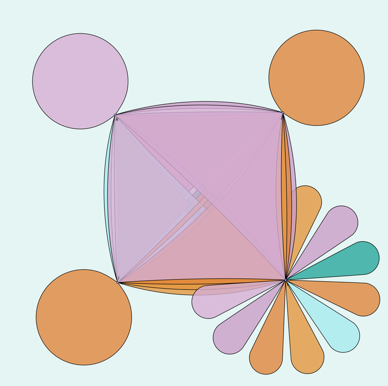


Figure 2: Compositional path closure.

Remark. Associativity depends on the specific choice of Δ and the edges E. There exist operators that guarantee it and others that break it.

Conclusions and future work

- The formalisation of compositions of n-ary relations provides a coherent framework for generating algebras from hypergraphs; it recovers the classical case and produces new structures.
- The thesis exhibits both associative and non-associative examples, and reproducible computational experiments are implemented in the Mathematica software.
- **Open direction:**
  1. Necessary and sufficient criteria for associativity (partial classification theorem).

References

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